Minimum Riesz s-energy problem

Let $S^{d-1} = \{ x \in \mathbb{R}^d : |x| = 1 \}$ be the unit sphere in $\mathbb{R}^d$, and $D_B = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1^2 + \cdots + x_d^2 \leq R^2 \}$ be the disk of radius $R$ in $\mathbb{R}^d$ with $d \geq 3$, and where $|.|$ is the Euclidean distance. The ring $s_a(b)$ in $\mathbb{R}^d$ is defined as $s_a(b) = \{ (r, \theta), r \leq a, \theta \leq \theta_b \in \mathbb{S}^{d-2} \}$, and the unit disk in $\mathbb{R}^2$ will be denoted by $D$. Given a compact set $E \subset \mathbb{D}$, consider the class $\mathcal{M}(E)$ of unit positive Borel measures supported on $E$. For $0 < s < d$, the Riesz $s$-potential and Riesz $s$-energy of a measure $\mu \in \mathcal{M}(E)$ are defined respectively as

$$U_s(\mu)(x) = \int \left| \frac{1}{|x-y|^s} - \frac{1}{|x|^s} \right| d\mu(y),$$

$$I_s(\mu) = \frac{1}{|E|} \int \left| \frac{1}{|x-y|^s} - \frac{1}{|x|^s} \right| d\mu(y).$$

Let $W_s(E) = \inf \{ I_s(\mu) : \mu \in \mathcal{M}(E) \}$. Define the Riesz $s$-capacity of $E$ as $\text{cap}_s(E) = 1/W_s(E)$. When $\text{cap}_s(E) > 0$, there is a unique $\mu_s(E)$ such that $I_s(\mu_s(E)) = W_s(E)$. Such $\mu_s$ is called the Riesz $s$-equilibrium measure for $E$.

An external field is defined as a continuous function $Q : E \rightarrow [0, \infty]$, such that $Q(x) < \infty$ on a set of positive surface area measure $S$. The weighted energy associated with $Q(x)$ is then defined by

$$I_s(\mu_Q) = I_s(\mu) + \int Q(x) d\mu(x).$$

The minimum problem in the presence of the external field $Q(x)$ refers to the minimal quantity $V_Q = \inf \{ I_s(\mu) : \mu \in \mathcal{M}(E) \}$.

A measure $\mu_0 \in \mathcal{M}(E)$ such that $I_s(\mu_0) = V_Q$ is called the s-extremal (or positive Riesz $s$-equilibrium) measure associated with $Q(x)$. The potential $U^0_s$ of the measure $\mu_0$ satisfies the Gauss variational inequalities

$$U^0_s(x) + Q(x) \geq F_0 \quad \text{on } E,$$

$$U^0_s(x) + Q(x) = F_0 \quad \text{for all } x \in S_Q,$$

where $F_0 = V_Q - \int Q(x) d\mu_0(x)$, and $S_Q := \text{supp} \mu_0$.

Sufficient conditions on an external field $Q(x)$ that guarantee that the support of the extremal measure $\mu_0$ is a ring or a disk

Theorem

Let $s = (d - 3) + 2\nu$, with $0 < \nu < 1$. Assume that an external field $Q : E \rightarrow [0, \infty]$ is invariant with respect to rotations about the polar axis, that is $Q(x) = Q(r)$, where $x = (0, r, \tau) \in D_B$, $\tau \in \mathbb{S}^{d-2}$, $0 \leq \tau \leq 1$. Furthermore, suppose that $Q$ is a convex function, that is $Q(r)$ is convex on $(0,1)$. Then the support of the extremal measure $\mu_0$ is a ring $\mathcal{R}(a,b)$, contained in the disk $D_B$. In other words, there exist real numbers $a$ and $b$ such that $0 \leq a < b \leq 1$, so that $\text{supp} \mu_0 = \mathcal{R}(a,b)$.

Furthermore, if $Q(r)$ is, in addition, an increasing function, then $\nu = 0$, which implies that the support of the extremal measure $\mu_0$ is a disk of radius $b < 1$, contained in the origin. On the other hand, if $Q(r)$ is a decreasing function, then $b = 1$, that is the support of the extremal measure $\mu_0$ will be a ring with outer radius 1.

Recovering of the extremal measure $\mu_0$

Theorem

Suppose that the support of the extremal measure $\mu_0$ is the disk $D_B$, and the external field $Q$ is invariant with respect to rotations about the polar axis, that is $Q(r) = Q(r)$, where $x = (0, r, \tau) \in D_B$, $\tau \in \mathbb{S}^{d-2}$, $0 \leq \tau \leq 1$. Also assume that $Q \in C^2(D_B)$. Let $s = (d - 3) + 2\nu$, with $0 < \nu < 1$, and let

$$F_t = \frac{\sin(\nu \lambda) \Gamma((d-3)/2+\lambda)}{\pi^{(d-2)/2} \Gamma(\lambda)} \ln \left( \frac{Q(u)}{r^{(s-2)/2}} \right) + \frac{\ln \left( \frac{Q(u)}{r^{(s-2)/2}} \right)}{r^{(s-2)/2}}, \quad 0 \leq r \leq R.$$

Then for the extremal measure $\mu_0$ we have

$$\text{d} \mu_0(x) = f(r) r^{s-1} dr \text{d} \mathcal{E}_s(x), \quad x = (0, r, \tau) \in D_B, \quad \tau \in \mathbb{S}^{d-2}, \quad 0 \leq r \leq R,$

where the density $f$ is explicitly given by

$$f(r) = C_0 (R^2 - r^2)^{s-1} + F_t(r), \quad 0 \leq r \leq R,$$

with the constant $C_0$ uniquely defined by

$$C_0 = \frac{1}{\Gamma(\nu \lambda) \Gamma((d-3)/2+\lambda)} \frac{\Gamma((d-1)/2)}{\pi^{(d-2)/2}}, \quad \int_0^R F_t(r) r^{s-2} dr.$$